

# THE RANGE OF A REAL SMIRNOV FUNCTION

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ABSTRACT. We give a characterization of the ranges of real Smirnov functions. In addition, we discuss the valence of such functions.

## 1. MAIN RESULT

In this paper we explore the possible ranges of a certain class of analytic functions on the open unit disk  $\mathbb{D}$  which are, in some sense, real on the unit circle  $\mathbb{T}$ . Informally, we know that the range of a (non-constant) analytic function on  $\mathbb{D}$  must be an open connected subset of  $\mathbb{C}$ . Since the functions we will be exploring in this paper have real boundary values (to be precisely defined in a moment), the range of such functions should involve slit-like domains where the slit is on the real axis.

The class of functions we are exploring here are a certain subclass of the *Smirnov class*  $N^+$  [5, Ch. 2] of analytic functions on  $\mathbb{D}$  which can be written as the quotient of two bounded analytic functions where the denominator is an outer function. Note that all of the classical Hardy spaces  $H^p$ ,  $p \in (0, \infty)$ , of  $f \in \mathcal{O}(\mathbb{D})$  (the analytic functions on  $\mathbb{D}$ ) for which the integral means

$$(1.1) \quad M_p(r, f) := \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

are uniformly bounded for  $r \in [0, 1)$ , are contained in  $N^+$  [5, Ch. 2].

Well-known classical theorems of Fatou and Riesz [5, Ch.1, 2] say that for each  $\varphi \in N^+$  the radial limit

$$(1.2) \quad \varphi(e^{i\theta}) := \lim_{r \rightarrow 1^-} \varphi(re^{i\theta})$$

exists for almost every  $\theta \in [0, 2\pi]$ . We say a function  $\varphi \in \mathcal{O}(\mathbb{D})$  belongs to the *real Smirnov class*  $N_{\mathbb{R}}^+$  if  $\varphi \in N^+$  and

$$\varphi(e^{i\theta}) \in \mathbb{R}$$

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for almost every  $\theta$  (see some examples below). These functions have been studied in [6, 7, 10, 11, 12] and a full characterization of them was given by Helson [10, 11] as

$$\varphi \in N_{\mathbb{R}}^+ \iff \varphi = i \frac{u+v}{u-v},$$

where  $u$  and  $v$  are inner functions and  $u-v$  is an outer function. Just as a reminder, an inner function is a bounded analytic function on  $\mathbb{D}$  whose radial boundary function from (1.2) is unimodular almost everywhere, while an outer function is an  $f \in N^+$  which takes the form

$$f(z) = c \exp \left( \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \psi(\theta) \frac{d\theta}{2\pi} \right), \quad \log \psi \in L^1[0, 2\pi], c > 0.$$

We will make use of the facts from [8, p. 65] and [8, p. 109] that if  $f \in \mathcal{O}(\mathbb{D})$  with  $\Re f > 0$ , then  $f \in H^p$  for all  $p \in (0, 1)$  and is an outer function. As an application of this we see that if  $g \in \mathcal{O}(\mathbb{D})$  with  $g(\mathbb{D}) \subset \mathbb{D}$ , then  $1+g$  is an outer function.

An analysis of unbounded symmetric Toeplitz operators from [10] (see also [6]) shows that when  $\varphi \in N_{\mathbb{R}}^+$ , the  $\mathbb{N}_0 \cup \{\infty\}$ -valued valence function

$$(1.3) \quad \lambda \mapsto \text{card}(\{z \in \mathbb{D} : \varphi(z) = \lambda\}), \quad \lambda \notin \mathbb{R},$$

is constant on each of the half planes

$$\mathbb{C}_+ := \{\Im z > 0\}, \quad \mathbb{C}_- := \{\Im z < 0\}.$$

Moreover [10, Thm. 4], given a pair  $(m, n)$ , with  $m, n \in \mathbb{N}_0 \cup \{\infty\}$ , one can find a  $\varphi \in N_{\mathbb{R}}^+$  such that the valence counting function from (1.3) is equal to  $m$  on  $\mathbb{C}_+$  and  $n$  on  $\mathbb{C}_-$ .

The main result of this paper is about the possible range of  $\varphi$ . Assuming that  $\varphi$  is not a (real) constant function, we know that  $\varphi(\mathbb{D})$  must be an open and connected subset of  $\mathbb{C}$ . What other restrictions must we have on  $\varphi(\mathbb{D})$ ? To give the reader a feel for where we are heading, let us consider a few examples of ranges of  $\varphi \in N_{\mathbb{R}}^+$ . For example, if

$$\varphi_1(z) = i \frac{1+z}{1-z},$$

the inverse of the standard Cayley transform  $z \mapsto \frac{z+i}{z-i}$  (which maps  $\mathbb{C}_+$  onto  $\mathbb{D}$ ), then  $\varphi_1 \in N^+$  (since it is the quotient of two bounded analytic functions and the denominator  $1-z$  is outer – see the comment above immediately following the definition of an outer function) and

$$\varphi_1(e^{i\theta}) = -\cot\left(\frac{\theta}{2}\right) \in \mathbb{R},$$

which says that  $\varphi_1 \in N_{\mathbb{R}}^+$ . Furthermore,  $\varphi_1(\mathbb{D}) = \mathbb{C}_+$ . In a similar way, we see that if

$$\varphi_2(z) = -i \frac{1+z}{1-z},$$

then  $\varphi_2 \in N_{\mathbb{R}}^+$  and  $\varphi_2(\mathbb{D}) = \mathbb{C}_-$ . If

$$\varphi_3(z) = \left( \frac{1+z}{1-z} \right)^4$$

then

$$\varphi_3(e^{i\theta}) = \cot^4\left(\frac{\theta}{2}\right) \in \mathbb{R}$$

and, since

$$z \mapsto \frac{1+z}{1-z}$$

maps  $\mathbb{D}$  onto the right-half plane  $\{z : \Re z > 0\}$ , then  $\varphi_3(\mathbb{D}) = \mathbb{C} \setminus \{0\}$ .

If

$$(1.4) \quad \varphi_4(z) = \frac{z}{(1-z)^2},$$

the well-known Koebe function, then

$$\varphi_4(e^{i\theta}) = -\frac{1}{2} \frac{1}{1 - \cos t} \in \mathbb{R}$$

and  $\varphi_4(\mathbb{D})$  is the single slit domain  $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$ . If

$$\varphi_5(z) = \frac{iz}{1-z^2},$$

then

$$\varphi_5(e^{i\theta}) = -\frac{1}{2} \csc \theta \in \mathbb{R}$$

and  $\varphi_5(\mathbb{D})$  turns out to be the double slit domain

$$\mathbb{C} \setminus ((-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty)).$$

(see Proposition 1.6 below for a proof.) From these examples, it seems like the range  $\varphi(\mathbb{D})$ , for  $\varphi \in N_{\mathbb{R}}^+$  (and not a real constant), can be equal to either  $\mathbb{C}_+$ ,  $\mathbb{C}_-$ , or certain domains of the form  $\mathbb{C} \setminus E$ , where  $E \subsetneq \mathbb{R}$  and closed.

Our main result about the range of  $\varphi \in N_{\mathbb{R}}^+$  is the following:

**Theorem 1.5.** *If  $\varphi \in N_{\mathbb{R}}^+$  and non-constant, then  $\varphi(\mathbb{D})$  is either  $\mathbb{C}_+$ ,  $\mathbb{C}_-$ , or  $\mathbb{C} \setminus E$  where  $E \subsetneq \mathbb{R}$  and closed. Moreover, given any closed  $E \subsetneq \mathbb{R}$ , there is a  $\varphi \in N_{\mathbb{R}}^+$  such that  $\varphi(\mathbb{D}) = \mathbb{C} \setminus E$ .*

To prove this theorem we first need to dispose of a special case, the double slit domain.

**Proposition 1.6.** *Given  $a, b \in \mathbb{R}$  with  $a < b$ , there is a  $\varphi \in N_{\mathbb{R}}^+$  whose range is  $\mathbb{C} \setminus ((-\infty, a] \cup [b, \infty))$ .*

*Proof.* The idea of this proof comes from a construction in [2, Thm. 4.5]. Consider the function

$$\varphi(z) = -i \left( \frac{1+z}{1-z} - \frac{1-z}{1+z} \right) = \frac{4iz}{z^2 - 1}.$$

Observe that  $\varphi \in N^+$  and a calculation will show that

$$\varphi(e^{i\theta}) = 2 \csc \theta,$$

and so  $\varphi \in N_{\mathbb{R}}^+$ . Now check that

$$\varphi(e^{i\theta}) > 0, \quad \theta \in (0, \pi), \quad \varphi(e^{i\theta}) < 0, \quad \theta \in (-\pi, 0);$$

$$\varphi(e^{i\theta}) \uparrow +\infty \text{ as } \theta \rightarrow 0^+, \quad \varphi(e^{i\theta}) \uparrow +\infty \text{ as } \theta \rightarrow \pi^-;$$

$$\varphi(e^{i\theta}) \downarrow -\infty \text{ as } \theta \rightarrow 0^-, \quad \varphi(e^{i\theta}) \downarrow -\infty \text{ as } \theta \rightarrow -\pi^+;$$

$$2 = \min\{\varphi(e^{i\theta}) : \theta \in (0, \pi)\};$$

$$-2 = \max\{\varphi(e^{i\theta}) : \theta \in (-\pi, 0)\}.$$

The function  $\varphi$  has valence at most 2 as a map from  $\mathbb{D}$  into the Riemann sphere and the above facts show that all of the values in  $(-\infty, -2)$  and  $(2, \infty)$  are taken by the function  $\varphi$  twice on  $\mathbb{T}$ . Explicit calculations show that  $\varphi$  takes on the values 2 and  $-2$  only at  $i$  and  $-i$ , respectively. Thus

$$\varphi(\mathbb{D}) \subset \mathbb{C} \setminus \{(-\infty, -2] \cup [2, \infty)\}.$$

Since  $\varphi(e^{i\theta})$  runs through the ray  $(-\infty, -2]$  once in each direction as  $\theta$  goes from  $-\pi$  to 0 (and similarly  $\varphi(e^{i\theta})$  runs through the ray  $[2, \infty)$  once in each direction as  $\theta$  goes from 0 to  $\pi$ ), we see from the argument principle that

$$\varphi(\mathbb{D}) = \mathbb{C} \setminus ((-\infty, -2) \cup (2, \infty)).$$

The function

$$z \mapsto -i \frac{b-a}{4} \left( \frac{1+z}{1-z} - \frac{1-z}{1+z} \right) + \frac{a+b}{2}$$

gives the required domain and range.  $\square$

One could also construct a function with the desired (double slit) properties by composing the map

$$z \mapsto \frac{bz + \frac{3}{4}a + \frac{1}{4}b}{z + 1}$$

with the Koebe function from (1.4). Note that the above map is the Möbius transform that sends  $-\frac{1}{4}$  to  $a$ ,  $-1$  to  $\infty$ , and  $\infty$  to  $b$ . It is not difficult to show that the resulting map belongs to  $H^p$  for  $p \in (0, \frac{1}{2})$  (and hence in the Smirnov class  $N^+$ ), and that it has the correct range and real boundary values.

*Proof of Theorem 1.5.* For the first part of the theorem, observe that clearly we can have  $\varphi(\mathbb{D}) = \mathbb{C}_+$  (e.g.,  $\varphi(z) = i(1+z)/(1-z)$ ) or  $\varphi(\mathbb{D}) = \mathbb{C}_-$  (e.g.,  $\varphi(z) = -i(1+z)/(1-z)$ ). If

$$\varphi(\mathbb{D}) \cap \mathbb{C}_+ \neq \emptyset \text{ and } \varphi(\mathbb{D}) \cap \mathbb{C}_- \neq \emptyset,$$

we can use the valence counting function from (1.3), and the fact that  $\varphi(\mathbb{D})$  is connected, to see that  $\varphi(\mathbb{D}) = \mathbb{C} \setminus E$  for some closed set  $E \subsetneq \mathbb{R}$ .

To prove the second part, let  $E \subsetneq \mathbb{R}$  be closed. For a moment, we will assume that  $E \neq \emptyset$ . We will take care of the case when  $E = \emptyset$ , i.e.,  $\varphi(\mathbb{D}) = \mathbb{C}$ , in Theorem 2.1 (in particular, Remark 2.2). In fact we will show a bit more.

Since  $E$  is a proper closed subset of  $\mathbb{R}$ , it follows that  $\mathbb{R} \setminus E$  contains at least one component. If it contains *exactly one* component, then  $E$  must be one of the following three options:

$$E = (-\infty, c], \quad E = [c, \infty), \quad E = (-\infty, a] \cup [b, \infty), \quad (a < b).$$

To find a  $\varphi \in N_{\mathbb{R}}^+$  with  $\varphi(\mathbb{D}) = \mathbb{C} \setminus E$  in the first two cases, use a simple variation of the Koebe function from (1.4). For the third case, the double slit domain, use Proposition 1.6.

Otherwise,  $\mathbb{R} \setminus E$  must have two components and, by means of a real translation of our function at the end, we can assume  $0 \in E$  and, for some  $a < 0$  and  $0 < b < c$ , that

$$(1.7) \quad E \subset (-\infty, a] \cup [0, b] \cup [c, \infty).$$

Define

$$E_+ = E \cap (0, \infty), \quad E_- = E \cap (-\infty, 0)$$

and the open set

$$(1.8) \quad \Omega = \{\Re z > 0\} \setminus (F_1 \cup F_2 \cup F_3),$$

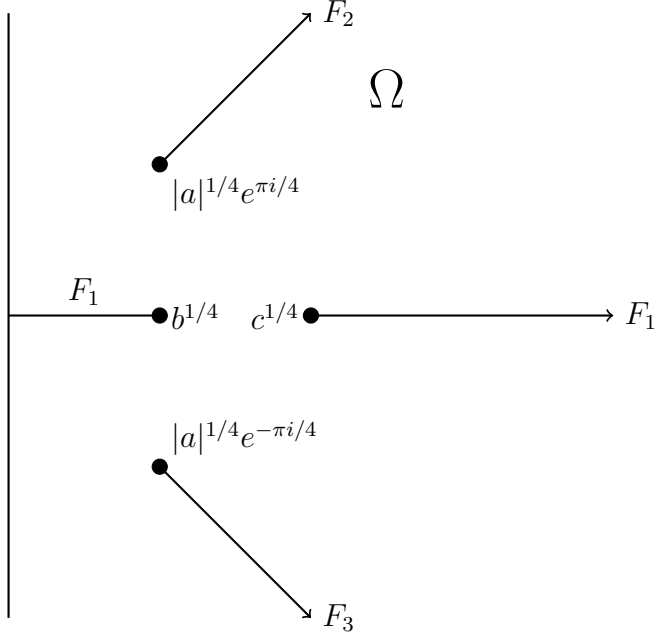


FIGURE 1. The region  $\Omega$  when  $E = (-\infty, a] \cup [0, b] \cup [c, \infty)$ .

where

$$\begin{aligned} F_1 &= \{x^{\frac{1}{4}} : x \in E_+\}, \\ F_2 &= e^{i\frac{\pi}{4}} \{(-x)^{\frac{1}{4}} : x \in E_-\}, \\ F_3 &= e^{-i\frac{\pi}{4}} \{(-x)^{\frac{1}{4}} : x \in E_-\}. \end{aligned}$$

See Figure 1 for an illustration of  $\Omega$  when

$$E = (-\infty, a] \cup [0, b] \cup [c, \infty).$$

Since  $\Omega$  is contained in

$$\{\Re z > 0\} \setminus \left( [0, b^{\frac{1}{4}}] \cup [c^{\frac{1}{4}}, \infty) \cup e^{i\pi/4} [a^{\frac{1}{4}}, \infty) \cup e^{-i\pi/4} [|a|^{\frac{1}{4}}, \infty) \right)$$

and this last set is connected, one concludes, also using the containment in (1.7), that  $\Omega$  is connected. By [4, p. 125] there exists a  $\psi \in \mathcal{O}(\mathbb{D})$ , an analytic covering map for  $\Omega$ , such that  $\psi(\mathbb{D}) = \Omega$ . Furthermore, as discussed earlier, since  $\psi(\mathbb{D})$  is contained in a half-plane, then  $\psi$  belongs to the Hardy space  $H^p$  for all  $p \in (0, 1)$  and so  $\psi \in N^+$ .

The map  $\psi$  is a covering map from  $\mathbb{D}$  to  $\Omega$ , which means that each point of  $\Omega$  is contained in an open neighborhood  $U$  such that  $\psi^{-1}(U)$  consists of disjoint open sets each of which is homeomorphic to  $U$  under  $\psi$ .

We now claim that if the radial limit

$$(1.9) \quad \lim_{r \rightarrow 1^-} \psi(re^{i\theta})$$

exists, which it will for almost every  $\theta$  (see (1.2)), then this value must belong to  $\partial\Omega$ . Indeed, if this were not the case, then the limit would be equal to some  $w \in \Omega$ . Now choose an open neighborhood  $U$  of  $w$  such that  $\psi^{-1}(U)$  consists of disjoint sets each of which is homeomorphic to  $U$  under  $\psi$ . Let  $W$  be an open neighborhood of  $w$  contained in  $U$  and such that  $\overline{W} \cap U$  is compact. Then

$$\psi^{-1}(W) = \bigcup_{a \in A} V_a,$$

where  $A$  is some index set and the  $V_a$  are pairwise disjoint open sets that are each homeomorphic to  $W$  under  $\psi$ . Thus each  $V_a$  has compact closure in  $\psi^{-1}(U)$ . For some  $b \in [0, 1)$ , the curve

$$r \rightarrow \psi([re^{i\theta}, e^{i\theta})), \quad r \in [b, 1),$$

must lie entirely in  $W$  since  $\psi$  has radial limit of  $w$  at  $e^{i\theta}$ . But since the  $V_a$  are disjoint open sets, this means that the ray

$$[re^{i\theta}, e^{i\theta}), \quad r \in [b, 1),$$

must lie entirely in one of the  $V_a$ . But this is impossible because each of the  $V_a$  has compact closure in  $\psi^{-1}(U)$  but  $e^{i\theta} \notin \psi^{-1}(U)$ .

Setting  $\varphi = \psi^4$  we see that  $\psi \in H^p$  for all  $p \in (0, \frac{1}{4})$  and thus  $\psi \in N^+$ . Moreover, since

$$\lim_{r \rightarrow 1^-} \psi(re^{i\theta}) \in \partial\Omega$$

for almost every  $\theta$ , we see that

$$\lim_{r \rightarrow 1^-} \varphi(re^{i\theta}) \in \mathbb{R}$$

for almost every  $\theta$ . Thus  $\varphi \in N_{\mathbb{R}}^+$ . The construction of  $\Omega$  from (1.8), and the fact that  $0 \in E$ , will show that

$$\varphi(\mathbb{D}) = \{z^4 : z \in \Omega\} = \mathbb{C} \setminus E. \quad \square$$

We remark that the mapping  $\varphi$  constructed above is actually outer. To see this, observe that  $\psi(\mathbb{D}) \subset \{\Re z > 0\}$ . As mentioned earlier, such functions are outer. Since the product of outer functions is another outer function, this means that  $\varphi = \psi^4$  is outer.

We also remark that the proof of Theorem 1.5, as well as Remark 2.2 and the proof of Theorem 2.1, show that we can find a  $\varphi$  with the desired properties that is in  $H^p$  for each  $p \in (0, \frac{1}{4})$ .

We observe that if the region  $\Omega$  occurring in the proof is not simply connected, then  $\varphi$  has infinite valence, since the fundamental group is infinite and thus the group of deck transformations of the universal cover is infinite [1]. If it is simply connected, then  $\varphi$  has valence two in its domain except at positive real numbers, where it has valence one.

## 2. CONTROLLING THE VALENCE

A key step in the construction of  $\varphi \in N_{\mathbb{R}}^+$  with  $\varphi(\mathbb{D}) = \mathbb{C} \setminus E$  was the uniformization theorem [4, p. 125]. However, it is not clear from our construction how one can control the valence of  $\varphi$ .

In this regard, one can ask the following question: Suppose we are given an closed set  $E \subsetneq \mathbb{R}$  and a pair  $(m, n)$ ,  $m, n \in \mathbb{N}_0 \cup \{\infty\}$ . Can we find a  $\varphi \in N_{\mathbb{R}}^+$  such that the valence of  $\varphi$  is equal to  $m$  on  $\mathbb{C}_+$ ,  $n$  on  $\mathbb{C}_-$ , and such that  $\varphi(\mathbb{D}) = \mathbb{C} \setminus E$ ? Can we say anything about the valence of  $\varphi$  on  $\mathbb{R} \setminus E$ ?

Our main result along these lines is the following:

**Theorem 2.1.** *Let  $n \geq 1$  and*

$$(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$$

*be a finite set of open intervals such that none of the intervals is the entire real line and  $(a_j, b_j)$  is disjoint from  $(a_{j+1}, b_{j+1})$  for each  $j$ . Then there is a real outer function whose range is*

$$\bigcup_{j=1}^n (a_j, b_j) \cup \mathbb{C}_+ \cup \mathbb{C}_-.$$

*Moreover, the valence of each point of  $\mathbb{C}_+$  is  $\lceil n/2 \rceil$  and the valence of each point of  $\mathbb{C}_-$  is  $\lfloor n/2 \rfloor$ . The valence of each point in  $\mathbb{R}$  is equal to the number of the intervals  $(a_j, b_j)$  in which it lies.*

Clearly we can interchange the roles of  $\mathbb{C}_+$  and  $\mathbb{C}_-$  in the above theorem.

**Remark 2.2.** As an application of Theorem 2.1, let  $n = 4$  and set the intervals to be

$$(0, \infty), \quad (-\infty, -1), \quad (2, 3), \quad (-2, 1).$$

Notice that each interval is disjoint from the one following it and their union is all of  $\mathbb{R}$ . Thus Theorem 2.1 produces a  $\varphi \in N_{\mathbb{R}}^+$  with  $\varphi(\mathbb{D}) = \mathbb{C}$ . This is the final case which needed to be resolved from the proof of Theorem 1.5.



The proof of Theorem 2.1 needs the following valence result.

**Lemma 2.3.** *Let  $f$  be an analytic function in  $\mathbb{D}$  of valence at most  $m$ . Then  $f \in H^p$  for every  $p \in (0, \frac{1}{2m})$ .*

*Proof.* Recall the definition of  $M_p(r, f)$  from (1.1) and define

$$M_\infty(r, f) := \sup_{|z|=r} |f(z)|.$$

By [3, Thm. 1] (see also [9, Sec. 2.3]) we have

$$(2.4) \quad M_\infty(r, f) = O\left(\frac{1}{(1-r)^{2m}}\right).$$

From [9, Theorem 3.2] we see that if  $f$  is  $m$ -valent and  $0 < r_0 < r < 1$  then

$$M_p(r, f) \leq M_\infty(r_0, f)^p + m \max\left(m, \frac{m^2}{2}\right) \int_{r_0}^r \frac{M_\infty(t, f)^p}{t} dt.$$

Applying the estimate in (2.4) for  $M_\infty(r, f)$  shows that the function  $r \mapsto M_p(r, f)$  is bounded when  $p \in (0, \frac{1}{2m})$ , i.e.,  $f \in H^p$  for all  $p \in (0, \frac{1}{2m})$ .  $\square$

*Proof of Theorem 2.1.* Construct a Riemann surface as follows. Weld a copy of  $\mathbb{C}_+$  to  $\mathbb{C}_-$  along the interval  $(a_1, b_1)$ . Now weld the copy of  $\mathbb{C}_-$  to a different copy of  $\mathbb{C}_+$  along the interval  $(a_2, b_2)$ . Now weld this copy of  $\mathbb{C}_+$  to a different copy of  $\mathbb{C}_-$  along the interval  $(a_3, b_3)$ . Proceed in this manner until all of the intervals are exhausted. Call this Riemann surface  $X$ . Let  $\theta$  be the projection map from  $X$  to  $\mathbb{C}$  that takes a given point in the Riemann surface to the corresponding point in either  $\mathbb{C}_+$ ,  $\mathbb{C}_-$ , or  $\mathbb{R}$ . See [1, II.3C] for more on welding Riemann surfaces.

We now claim that the Riemann surface is conformally equivalent to  $\mathbb{D}$ . If it were not, then, since it is simply connected, it would be equivalent to either the Riemann sphere or the complex plane (uniformization theorem). It is not equivalent to the sphere since it is not compact. Suppose it were equivalent to the plane. Let  $\psi$  be the conformal map from the plane to  $X$  and define  $g = \theta \circ \psi$ .

If  $g$  were univalent, then  $\theta$  would be as well, which would mean that  $X$  would have only one copy of  $\mathbb{C}_+$  and only one copy of  $\mathbb{C}_-$ . But then  $X$  would be a simply connected proper subset of  $\mathbb{C}$  and thus equivalent to  $\mathbb{D}$  and not the plane. Thus  $g$  is not univalent.

Note that  $g$  has valence at each point of at most  $\lceil n/2 \rceil$  and that it is locally univalent. Since  $g$  is an entire function of finite valence, we

can use Picard's theorem to see that  $g$  must be a polynomial. Now since  $\psi$  and  $\theta$  are locally univalent,  $g$  must be as well. But the only locally univalent polynomials are linear. Any locally univalent linear function is univalent and so  $g$  must be univalent. However, we already eliminated this possibility earlier. Thus  $X$  is not equivalent to  $\mathbb{C}$ .

Let  $\varphi$  be a map from  $\mathbb{D}$  to the Riemann surface and  $f = \theta \circ \varphi$ . Then  $f$  maps from  $\mathbb{D}$  into  $\mathbb{C}$ . Since  $f$  has valence at most  $\lceil n/2 \rceil$  at each point, it belongs to some Hardy space (Lemma 2.3) and thus belongs to  $N^+$ .

We will now show that  $f$  has real (radial) boundary values almost everywhere. To see this, note that every point in  $\mathbb{C} \setminus \mathbb{R}$  has a neighborhood  $U$  such that  $\theta^{-1}(U)$  consists of disjoint sets that are each homeomorphic to  $U$  under  $\theta$ . Thus every point in  $\mathbb{C} \setminus \mathbb{R}$  has a neighborhood  $U$  such that  $f^{-1}(U)$  consists of disjoint sets that are each homeomorphic to  $U$  under  $f$ . The same argument as used to prove (1.9) shows that no (radial) boundary values of  $f$  lie in  $\mathbb{C} \setminus \mathbb{R}$ .  $\square$

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